

# SMOOTH ACTIONS OF $Aff^+(\mathbb{R})$ ON COMPACT SURFACES WITH NO FIXED POINT: AN ELEMENTARY CONSTRUCTION

F.J. TURIEL

ABSTRACT. Any compact surface supports a continuous action of the orientation preserving affine group of the real line which is fixed point free (Lima and Plante). It is generally admitted that this action can be taken smooth although it is not easy finding references for it. Here one gives a such action.

## 1. INTRODUCTION

All structures and objects considered are smooth that is real  $C^\infty$ , unless another thing is stated. Manifolds can have nonempty boundary.

A classical theorem by Lima [4] states that any continuous action of  $\mathbb{R}^n$  on a compact surface of nonzero Eule-Poincaré characteristic possesses fixed points. Later on Plante [5] extended this result to connected nilpotent Lie groups (a short proof in class  $C^\infty$  of these results can be seen in [6]), Hirsch and Weinstein [2] to analytic actions of connected super-solvable Lie Groups and finally Hirsch [3] to local nilpotent actions.

Besides Lima and Plante proved that that every compact surface supports a continuous fixed point free action by the (orientation preserving) affine group  $Aff^+(\mathbb{R})$ , the first non-nilpotent Lie group. It belongs to the folklore that any compact surface (with or without boundary) supports a smooth action of  $Aff^+(\mathbb{R})$  fixed point free. Although this fact can be deduced from other results on surfaces (for instance see [1]), it seems useful providing an elementary construction of such kind of actions. That is the aim of this work.

For this purpose, if  $M$  is a compact surface it suffices finding two vector fields  $X$  and  $Y$  on it such that:

- (a)  $[X, Y] = -Y$ .
- (b)  $X$  and  $Y$  are tangent to the boundary of  $M$  if any.

(c) There is no common zero of  $X$  and  $Y$ .

## 2. TWO LEMMAS

**Lemma 1.** *For each natural  $k > 0$  there exist real numbers  $-1 < a_1 < \dots < a_k < 1$  and two smooth functions  $\varphi, \psi: \mathbb{R} \rightarrow \mathbb{R}$ , where  $\mathbb{R}$  is equipped with the variable  $t$ , such that:*

- (1)  $[X, Y] = -Y$  where  $X = \varphi \frac{\partial}{\partial t}$  and  $Y = \psi \frac{\partial}{\partial t}$ .
- (2)  $\psi^{-1}(0) = \{a_1, \dots, a_k\}$  and  $\varphi(t) = -(t - a_j)$  near each  $a_j$ .
- (3)  $\varphi = t$  outside  $(-1, 1)$ .

*Proof.* On  $\mathbb{R}$  consider the vector fields  $A = (1 + \cos t) \frac{\partial}{\partial t}$ ,  $B = -\sin t \frac{\partial}{\partial t}$  and  $C = (1 - \cos t) \frac{\partial}{\partial t}$ ; then  $[A, B] = -A$ ,  $[B, C] = -C$ . Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function such that:

$$f(t) = 0 \text{ if } |t| \geq 4k\pi,$$

$$f > 0 \text{ on } (-2\pi, 2k\pi),$$

$$f = 1 + \cos t \text{ on } (2\ell\pi - \varepsilon, 2\ell\pi + \varepsilon), \ell = 0, \dots, k-1, \text{ for some } \varepsilon > 0,$$

$$f = 1 - \cos t \text{ on } (-2\pi, -\pi] \cup [(2k-1)\pi, 2k\pi).$$

The vector field  $Z = f \frac{\partial}{\partial t}$  is complete and its integral curve, passing through  $-\pi$  for the time zero, runs from  $-2\pi$  to  $2k\pi$ . So there exists a diffeomorphism  $F: (-2\pi, 2k\pi) \rightarrow \mathbb{R}$  such that  $F_*Z = \frac{\partial}{\partial t}$  and  $F(-\pi) = 0$ .

Set  $\tilde{B} = F_*B$  and  $\tilde{C} = F_*C$ . Then  $\tilde{C}$  only vanishes at  $F(\{0, 2\pi, \dots, 2(k-1)\pi\})$ ,  $\tilde{B} = t \frac{\partial}{\partial t}$  on  $(-\infty, 0]$ ,  $\tilde{B} = (t-a) \frac{\partial}{\partial t}$  on  $[a, \infty)$  where  $a = F((2k-1)\pi)$  and  $\tilde{B} = -(t - F(2\ell\pi)) \frac{\partial}{\partial t}$  near each  $F(2\ell\pi)$ ,  $\ell = 0, \dots, k-1$ .

Indeed, as  $Z$  equals  $C$  on  $(-2\pi, -\pi] \cup [(2k-1)\pi, 2k\pi)$  then  $[\frac{\partial}{\partial t}, \tilde{B}] = \frac{\partial}{\partial t}$  on the image under  $F$  of this set, which implies  $\tilde{B} = t \frac{\partial}{\partial t}$  on  $(-\infty, 0] = F((-\infty, -\pi])$  since  $\tilde{B}(0) = F_*(B(-\pi)) = 0$  and  $\tilde{B} = (t-a) \frac{\partial}{\partial t}$  on  $[a, \infty) = F([(2k-1)\pi, 2k\pi))$  because  $\tilde{B}(a) = F_*(B((2k-1)\pi)) = 0$ . By a similar reason as  $Z$  equals  $A$  near  $2\ell\pi$ ,  $\ell = 0, \dots, k-1$ , then  $[\frac{\partial}{\partial t}, \tilde{B}] = -\frac{\partial}{\partial t}$  and  $\tilde{B} = -(t - F(2\ell\pi)) \frac{\partial}{\partial t}$  close to  $F(2\ell\pi)$ ,  $\ell = 0, \dots, k-1$ .

Finally, consider a diffeomorphism  $G: \mathbb{R} \rightarrow \mathbb{R}$  equals to  $Id$  on  $(-\infty, 0]$ , to  $(t-a)$  on  $[a+1, \infty)$  and to a translation near each  $F(2\ell\pi)$ ,  $\ell = 0, \dots, k-1$ , and set  $X = G_*\tilde{B}$ ,  $Y = G_*\tilde{C}$  and  $a_j = G(F(2(j-1)))$ ,  $j = 1, \dots, k$ .  $\square$

Endow  $\mathbb{R} \times S^1$  with coordinates  $(t, \theta)$ .

**Lemma 2.** *Consider a scalar  $c \neq 0$  and a vector field  $Y$  on  $(-\varepsilon, 0) \times S^1$  such that  $[X, Y] = -Y$  where  $X = t^2 e^{1/t} \frac{\partial}{\partial t}$ . Then there exists a diffeomorphism  $G: (-\varepsilon, 0) \times S^1 \rightarrow (-\varepsilon, 0) \times S^1$  and two vector fields  $\tilde{X}, \tilde{Y}$  on  $(-\varepsilon, \infty) \times S^1$  such that  $G_*X = \tilde{X}$ ,  $G_*Y = \tilde{Y}$  on  $(-\varepsilon, 0) \times S^1$ , and  $\tilde{X} = c \frac{\partial}{\partial \theta}$ ,  $\tilde{Y} = 0$  on  $[0, \infty) \times S^1$  (which implies  $[\tilde{X}, \tilde{Y}] = -\tilde{Y}$  everywhere).*

*Moreover  $G$  preserves the orientation.*

*Proof.* Since  $(t^2 e^{1/t} \frac{\partial}{\partial t})e^{-1/t} = 1$  and  $[X, \frac{\partial}{\partial \theta}] = 0$  one has

$$Y = \exp(-e^{-1/t})f_1(\theta)X + \exp(-e^{-1/t})f_2(\theta)\frac{\partial}{\partial \theta}.$$

Now consider the diffeomorphism  $G: (-\varepsilon, 0) \times S^1 \rightarrow (-\varepsilon, 0) \times S^1$  given by  $G(t, \theta) = (t, \theta + \pi(ce^{-1/t}))$ , where  $\pi: \mathbb{R} \rightarrow S^1 \equiv \mathbb{R}/2\pi\mathbb{Z}$  is the canonical projection. Then  $G_*X = t^2 e^{1/t} \frac{\partial}{\partial t} + c \frac{\partial}{\partial \theta}$ , which is smoothly prolonged to a vector field  $\tilde{X}$  on  $(-\varepsilon, \infty) \times S^1$  by setting  $\tilde{X} = c \frac{\partial}{\partial \theta}$  on  $[0, \infty) \times S^1$ . On the other hand

$$G_*Y = \exp(-e^{-1/t})f_1(\theta - \pi(ce^{-1/t}))\tilde{X} + \exp(-e^{-1/t})f_2(\theta - \pi(ce^{-1/t}))\frac{\partial}{\partial \theta}$$

that prolongs by zero to a smooth vector field  $\tilde{Y}$  on  $(-\varepsilon, \infty) \times S^1$ .

Indeed, given  $g: S^1 \rightarrow \mathbb{R}$  set  $g^{(\ell)} = \frac{\partial^\ell g}{\partial \theta^\ell}$ . For every  $k, \ell \in \mathbb{N}$  the function  $g^{(\ell)}(\theta - \pi(c/s))$  is dominated on  $\mathbb{R}^+ \times S^1$  by  $s^{-k}e^{-1/s}$  because  $g^{(\ell)}(\theta - \pi(c/s))$  is bounded on this set, which implies that the function

$$\lambda(s, \theta) = \begin{cases} e^{-1/s}g(\theta - \pi(c/s)) & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}$$

is smooth on  $\mathbb{R} \times S^1$ . Therefore (set  $s = e^{1/t}$  if  $t < 0$  and  $s = 0$  if  $t \geq 0$ ) the functions  $\tilde{f}_k$ ,  $k = 1, 2$ , given by

$$\tilde{f}_k(t, \theta) = \begin{cases} \exp(-e^{-1/t})f_k(\theta - \pi(ce^{-1/t})) & \text{if } t < 0 \\ 0 & \text{if } t \geq 0 \end{cases}$$

are smooth on  $(-\varepsilon, \infty) \times S^1$ . □

Consider a surface  $S$  equipped with two vector fields  $X_S, Y_S$ . A *neat hole* of parameter  $c \neq 0$  (on  $S$  relative to  $X_S, Y_S$ ) is an open set  $A$  of

$S$  identified to the cylinder  $(0, \infty) \times S^1$  by means of a diffeomorphism  $F: A \rightarrow (0, \infty) \times S^1$  in such a way that:

- (1) The "side"  $\{\infty\} \times S^1$  of the cylinder defines an end of  $S$ .
- (2)  $F^{-1}: (0, \infty) \times S^1 \rightarrow A \subset S$  extends into a continuous map from  $[0, \infty) \times S^1$  to  $S$ .
- (3)  $X_S, Y_S$  regarded on  $(0, \infty) \times S^1$  equal the vector fields  $\tilde{X}, \tilde{Y}$  of Lemma 2, that is  $F_*X_S = c \frac{\partial}{\partial \theta}$  and  $F_*Y_S = 0$ .

*Truncating a neat hole* means removing the set  $(b, \infty) \times S^1$ , for some  $b > 0$ , from  $A \equiv (0, \infty) \times S^1$ , so from  $S$ . Therefore the neat hole becomes a collar of a  $S^1$ -component of the boundary, to which  $X_S, Y_S$  are tangent.

### 3. NEAT HOLES ON $\mathbb{R}^2$

On  $\mathbb{R}^2$ , with coordinates  $(x_1, x_2)$ , when  $n \geq 1$  one considers the vector fields  $X = \varphi_1(x_1) \frac{\partial}{\partial x_1} + \varphi_2(x_2) \frac{\partial}{\partial x_2}$  and  $Y = \psi_1(x_1) \frac{\partial}{\partial x_1} + \psi_2(x_2) \frac{\partial}{\partial x_2}$  where  $\varphi_1, \psi_1$  are like in Lemma 1 for  $k = 1$ , and  $\varphi_2, \psi_2$  are like in Lemma 1 for  $k = n$ ; set  $\psi_1^{-1}(0) = \{a_1\}$  and  $\psi_2^{-1}(0) = \{b_1, \dots, b_n\}$ . On the other hand if  $n = 0$  we consider  $X = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$  and  $Y = \frac{\partial}{\partial x_1}$ . Obviously  $[X, Y] = -Y$  and  $X$  is complete because (3) of Lemma 1; moreover the common zeros of  $X, Y$  are  $(a_1, b_1), \dots, (a_1, b_n)$  if any.

*Constructing neat holes at  $(a_1, b_1), \dots, (a_1, b_n)$ .*

Up to translation it is enough to consider the case where  $(a_1, b_j) = (0, 0)$ ,  $X = -x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}$  and  $Y = \psi_1(x_1) \frac{\partial}{\partial x_1} + \psi_2(x_2) \frac{\partial}{\partial x_2}$  near the origin.

For a suitable  $d > 0$  identify  $(d, \infty) \times S^1$ , endowed with coordinates  $(r, \theta)$ , with a sufficiently small punctured ball  $B_\rho(0) - \{0\}$  by setting  $x_1 = r^{-1} \cos \theta$ ,  $x_2 = r^{-1} \sin \theta$ . Then  $X = r \frac{\partial}{\partial r}$ .

Let  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function such that, for some  $\varepsilon > 0$ :

$$\varphi(t) = \begin{cases} t+1 & \text{if } -1-\varepsilon < t < -1+\varepsilon \\ t^2 e^{1/t} & \text{if } -\varepsilon < t < 0 \\ > 0 & \text{if } -1 < t < 0 \\ 0 & \text{if } t \geq 0. \end{cases}$$

Since  $\varphi \frac{\partial}{\partial t}$  on  $(-1, 0)$  and  $r \frac{\partial}{\partial r}$  on  $\mathbb{R}^+$  are complete and never vanish there exists a diffeomorphism  $f: \mathbb{R}^+ \rightarrow (-1, 0)$  such that  $f_* \left( r \frac{\partial}{\partial r} \right) = \varphi \frac{\partial}{\partial t}$ . That gives rise to a diffeomorphism  $F: (r, \theta) \in \mathbb{R}^+ \times S^1 \rightarrow (f(r), \theta) \in (-1, 0) \times S^1$  which transforms  $r \frac{\partial}{\partial r}$  into  $\varphi \frac{\partial}{\partial t}$ .

Thus for a suitable  $\varepsilon > 0$  the punctured ball  $B_\rho(0) - \{0\}$  can be identified with  $(-\varepsilon, 0) \times S^1$ , in such a way that  $X = t^2 e^{1/t} \frac{\partial}{\partial t}$ . By Lemma 2, attaching  $(-\varepsilon, \infty) \times S^1$  to  $\mathbb{R}^2 - \{(a_1, b_1), \dots, (a_1, b_n)\}$  around  $(a_1, b_j)$  in the obvious way (that is by means of the diffeomorphism  $G$  given by this result), provides us with a neat hole of parameter any  $c \neq 0$ .

*The neat hole at the infinity of  $\mathbb{R}^2$ .*

As  $\varphi_1$  and  $\varphi_2$  are bounded on  $[-1, 1]$  then  $x_1 \varphi_1(x_1) + x_2 \varphi_2(x_2) \geq 1$  if  $\|x\| \geq \rho$  for some  $\rho$  big enough (for the case  $n = 0$  it is obvious). In other words  $X$  is outwardly transverse to any sphere  $S_\lambda^1$ ,  $\lambda \geq \rho$ , and runs to infinity. This allows us to identify  $\mathbb{R}^2 - D_\rho^2$  with  $\mathbb{R}^+ \times S^1$ , endowed with coordinates  $(\tilde{r}, \theta)$ , by means of the integral curves of  $X$  passing through  $S_\rho^1$  for the time zero. Now  $X = \frac{\partial}{\partial \tilde{r}}$ .

Finally setting  $r = e^{\tilde{r}}$  we have  $X = r \frac{\partial}{\partial r}$  on  $(1, \infty) \times S^1$  with coordinates  $(r, \theta)$ . The remainder is the same as before.

#### 4. CONSTRUCTION OF THE ACTION OF $Aff^+(\mathbb{R})$

The construction above of a neat hole means attaching a cylinder  $(-\varepsilon, \infty) \times S^1$ ; however the space remains diffeomorphic to  $\mathbb{R}^2 - \{(a_1, b_1), \dots, (a_1, b_n)\}$ ,  $n \geq 0$ , that is to  $S^2$  with  $n + 1$  holes, but now the vector fields on it are easily managed.

On  $[0, \pi] \times \mathbb{R}$  endowed with coordinates  $(y_1, y_2)$  consider the vector fields  $X' = c \frac{\partial}{\partial y_1}$ ,  $c \in \mathbb{R} - \{0\}$ , and  $Y' = 0$ . Now identifying each  $(0, y_2)$  with  $(\pi, -y_2)$  gives rise to the open Moebius strip  $M$  equipped with two vector fields  $X$  and  $Y$  with no common zero such that  $[X, Y] = -Y$ . Moreover  $(M - N, X, Y)$ , where  $N$  is given by the condition  $y_2 = 0$  (before identifying), is (diffeomorphic to) a neat hole of parameter  $c$ .

On the other hand two different neat holes, belonging to the same surface or not, can be gluing together under the diffeomorphism  $(t, \theta) \in \mathbb{R}^+ \times S^1 \rightarrow (t^{-1}, h(\theta)) \in \mathbb{R}^+ \times S^1$  where  $h$  belongs to the orthogonal group  $O(2)$ . When

the parameter of each neat hole may be independently chosen, it is always possible at the same time, by gluing the vector fields also, to construct two vector fields  $\overline{X}, \overline{Y}$  with no common zero such that  $[\overline{X}, \overline{Y}] = -\overline{Y}$ .

Finally, considering  $S^2$  with  $n + 1$  neat holes constructed as before for a suitable  $n$ , truncating some neat holes (only if the boundary is nonempty), gluing together or not some pairs of neat holes (perhaps some of them reversing the orientation), gluing or not an open disk (that is  $\mathbb{R}^2$  with the neat hole at the infinity) or an open Moebius strip, leads to construct two vector fields  $\widehat{X}, \widehat{Y}$  on any connected compact surface, with no common zero and tangent to the boundary if any, such that  $[\widehat{X}, \widehat{Y}] = -\widehat{Y}$ .

Therefore an action of  $Aff^+(\mathbb{R})$  with no fixed point can be constructed on any compact surface.

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(F.J. Turiel) GEOMETRÍA Y TOPOLOGÍA, FACULTAD DE CIENCIAS, CAMPUS DE TEATINOS, S/N, 29071-MÁLAGA, SPAIN

E-mail address: turiel@uma.es